PRACE NAUKOWE UNIWERSYTETU ŚLĄSKIEGO W KATOWICACH NR 2
PRACE MATEMATYCZNE I, 1969

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Remarks on differential concomitants of the covariant tensor

INTRODUCTION. Let X^n be an n-dimensional manifold. If the transformation of the coordinate system has the form

(1) $$\tilde{x}^i = \tilde{x}^i(x^k), \quad i, k = 1, 2, \ldots, n,$$

then we put

(2) $$A^i_k = \frac{\partial \tilde{x}^i}{\partial x^k}.$$

The determinant $J = \mathrm{Det}\,||A^i_k||$ is different from zero. For the inverse transformation to (1)

(3) $$x^i = x^i(\tilde{x}^k)$$

we put

(4) $$B^i_k = \frac{\partial x^i}{\partial \tilde{x}^k}.$$

The partial derivatives of a value U with respect to x^i will be denoted by $U_{,j} = \dfrac{\partial U}{\partial x^j}$.

We consider a symmetric tensor field g_{ij} of rank n on X^n. We say that a geometric object Θ is a differential concomitant of order s of the tensor g_{ij}, if for every coordinate systems (x^i) we have

(5) $$\Theta = \Theta(g_{ij}, g_{ij,k}, \ldots, g_{ij,k_1,\ldots,k_s})$$

(cf. [2], p. 148, also [6], p. 138).

In the present paper we consider the differential concomitants of the first and second order of g_{ij}, which are purely differential geometric object of the first class. After a change of the coordinate system (1) the components ω of these objects are transformed according to the rule

(6) $$\tilde{\omega} = F(\omega, A)$$

where $A = ||A^i_k|| \in Gl(n)$ and the function F satisfies the following equations

(7) $$\begin{aligned} &a) \quad F[F(\omega, A_1), A_2] = F(\omega, A_2 A_1), \\ &b) \quad F(\omega, E) = \omega. \end{aligned}$$

A_1, A_2 denote here arbitrary elements of the group $Gl(n)$, E is the unit element of the group $Gl(n)$ and A_2A_1 denotes the product of the matrices A_2, A_1.

In § 1 we show that such differential concomitants of the first order do not exist in the sense that every such a concomitant is an algebraic concomitant of the tensor g_{ij}.

In § 2 we show that every concomitant of the second order is an algebraic concomitant of the tensor g_{ij} and of the curvature tensor R_{ijkl}.

In § 3 we determine for $n=2$ the general form of those differential concomitants of the second order of g_{ij} which are scalars and W-densities.

In a next paper we shall determine such concomitants for the case $n=3$.

J. A. SCHOUTEN has proved in [6] (p. 138) that all differential concomitants of order s of the tensor g_{ij} are algebraic concomitants of g_{ij}, of the curvature tensor and of its covariant derivatives. In this way our result of § 1 and § 2 is a particular case of J. A. SCHOUTEN'S theorem. However, we give here its proof in the case $s=1$ and $s=2$, because in fact our proof is different from that given by J. A. SCHOUTEN.

M. A. MCKIERNAN and H. RICHARDS have obtained the same result with the aid of similar methods (cf. [4]).

§ 1. CASE $s=1$. If we denote by $\bar{g}_{ij}$ the components of the tensor g_{ij} in the system $(\bar{x}^i)$, then the transformation formula of these components has the following form

$$\bar{g}_{ij} = B_i^s B_k^t g_{st}. \tag{1.1}$$

We prove the following theorem:

THEOREM 1. *If a purely differential geometric object of the first class is a differential concomitant of the first orde· of a symmetric and regular tensor g_{ij}, then this object is an algebraic concomitant of the tensor g_{ij}.*

Proof. If an object ω is a differential concomitant of the first order of the tensor g_{ij}, then it must satisfy the following equation

$$\omega(\bar{g}_{ij}, \bar{g}_{ij,k}) = F[\omega(g_{ij}, g_{ij,k}), A], \tag{1.2}$$

where

$$\bar{g}_{ij,k} = B_{ik}^s B_j^t g_{st} + B_i^s B_{jk}^t g_{st} + B_i^s B_j^t B_k^r g_{st,r} \tag{1.3}$$

and

$$B_{ij}^s = \frac{\partial^2 x^s}{\partial \bar{x}^i \partial \bar{x}^j}.$$

We put $A=E$, then $B=||B_k^i||=E$. After this substitution the relations (1.1) and (1.3) have the forms

$$\bar{g}_{ij} = g_{ij}, \tag{1.4}$$

$$\bar{g}_{ij,k} = B_{ik}^s g_{sj} + B_{jk}^s g_{is} + g_{ij,k}. \tag{1.5}$$

Now we shall seek such values B^s_{ik} which satisfy the system of equations

$$(1.6)\qquad \begin{aligned} &a)\quad B^s_{ik}\, g_{sj} + B^s_{jk}\, g_{is} = -g_{ij,k},\\ &b)\quad B^s_{ik} = B^s_{kl}. \end{aligned}$$

It is known that the solution of this system of equations is

$$(1.7)\qquad B^s_{ik} = -\Gamma^s_{ik},$$

where Γ^s_{ik} denote the Christiffel symbols (cf. [2], p. 242). If we substitute (1.7) into (1.5), then we get

$$(1.8)\qquad \tilde{g}_{ij,k} = -\Gamma^s_{ik}\, g_{sj} - \Gamma^s_{jk} g_{is} + g_{ij,k} = 0,$$

since the tensor g_{ij} is covariant constant.

Let us substitute $A = E$, $B^s_{jk} = -\Gamma^s_{jk}$ into equation (1.2). Thus we have by (7*b*), (1.4) and (1.8)

$$\omega(g_{ij}, 0) = F[\omega(g_{ij}, g_{ij,k}), E] = \omega(g_{ij}, g_{ij,k}).$$

Therefore

$$(1.9)\qquad \omega(g_{ij}, g_{ij,k}) = f(g_{ij}).$$

This completes the proof.

§ 2. Case $s=2$. We denote by R_{ijkl} the covariant curvature tensor

$$(2.1)\qquad R_{ijkl} = 2g_{is}(\Gamma^s_{j[k,l]} + \Gamma^t_{j[k}\Gamma^s_{|t|l]}).$$

In this case we shall prove the following theorem:

Theorem 2. *If a purely differential geometric object of the first class is a differential concomitant of the second order of a symmetric and regular tensor g_{ij}, then this object is an algebraic concomitant of the tensors g_{ij} and R_{ijkl}.*

Proof. If the object ω is a differential concomitant of the second order of the tensor g_{ij}, then it must satisfy the following equation

$$(2.2)\qquad \omega(\tilde{g}_{ij}, \tilde{g}_{ij,k}, \tilde{g}_{ij,k,l}) = F[\omega(g_{ij}, g_{ij,l}, g_{ij,l,k}), A],$$

where

$$(2.3)\qquad \begin{aligned} \tilde{g}_{ij,k,l} = {} & B^s_{ikl} B^t_j g_{st} + B^s_{ik} B^t_{jl} g_{st} + B^s_{il} B^t_{jk} g_{st}\\ & + B^s_i B^t_{jkl} g_{st} + B^s_{ik} B^t_j B^r_l\, g_{st,r} + B^s_i B^t_{jk} B^r_l\, g_{st,r}\\ & + B^s_{il} B^t_j B^r_k\, g_{st,r} + B^s_i B^t_{jl} B^r_k\, g_{st,r}\\ & + B^s_i B^t_j B^r_{kl}\, g_{st,r} + B^s_i B^t_j B^r_k B^n_l\, g_{st,r,n} \end{aligned}$$

and

$$B^s_{ikl} = \frac{\partial^3 x^s}{\partial \bar{x}^i \partial \bar{x}^k \partial \bar{x}^l}.$$

We put $B = ||B^i_k|| = E$, $B^s_{jl} = -\Gamma^s_{jl}$ into (2.3). Relation (2.3) will have the following form

$$\tilde{g}_{ij,k,l} = B^s_{jkl}\, g_{is} + B^s_{ilk}\, g_{sj} - g_{is}(\Gamma^t_{jk}\Gamma^s_{lt} + \Gamma^t_{lk}\Gamma^s_{jt} - \Gamma^s_{jl,k}) - \\ - g_{sj}(\Gamma^t_{ik}\Gamma^s_{lt} + \Gamma^t_{lk}\Gamma^s_{it} - \Gamma^s_{il,k}). \tag{2.4}$$

We consider the following equation

$$B^s_{jkl}\, g_{is} + B^s_{ikl}\, g_{sj} = (\Gamma^t_{jk}\Gamma^s_{lt} + \Gamma^t_{lk}\Gamma^s_{jt} - \Gamma^s_{jl,k})\, g_{si} \\ + (\Gamma^t_{ik}\Gamma^s_{lt} + \Gamma^t_{lk}\Gamma^s_{it} - \Gamma^s_{il,k})\, g_{sj}. \tag{2.5}$$

The values

$$B^s_{jkl} = \Gamma^t_{jk}\Gamma^s_{lt} + \Gamma^t_{lk}\Gamma^s_{jt} - \Gamma^s_{jl,k} \tag{2.6}$$

fulfil equation (2.5). The values (2.6) are symmetric only with respect to the indexes j and l. We set

$$B^s_{jkl} = B^s_{klj} = B^s_{ljk} = B^s_{jlk} = B^s_{lkj} = B^s_{kjl} = \Gamma^t_{jk}\Gamma^s_{lt} + \Gamma^t_{lk}\Gamma^s_{jt} - \Gamma^s_{jl,k} \tag{2.7}$$

into (2.4). Then for all possible permutations of j, k, l, expression (2.4) will take on the following values:

$$\begin{aligned} \tilde{g}_{ij,k,l} &= 0 \\ \tilde{g}_{ik,l,j} &= R_{ijkl} \\ \tilde{g}_{il,j,k} &= R_{ilkj} \\ \tilde{g}_{ij,l,k} &= 0 \\ \tilde{g}_{il,k,j} &= R_{iljk} \\ \tilde{g}_{ik,j,l} &= R_{ijlk} \end{aligned} \tag{2.8}$$

We substitute $||A^i_j|| = ||B^i_j|| = E$, $B^i_{jk} = -\Gamma^i_{jk}$ and the expression given by (2.6) into equation (2.2). Thus we obtain by ($7b$), (1.4), (1.8) and (2.8)

$$\omega(g_{ij}, g_{ij,k}, g_{ij,k,l}) = \omega(g_{ij}, 0, 0, R_{ijkl}, R_{ilkj}, 0, R_{iljk}, R_{ijlk}).$$

Therefore ω depends only on the tensors g_{ij} and R_{ijkl}:

$$\omega(g_{ij}, g_{ij,k}, g_{ij,k,l}) = h(g_{ij}, R_{ijkl}).$$

This completes the proof.

§ 3. Now we consider a two-dimensional manifold X^2. We are going to determine the general form of those differential concomitants of the second order of the tensor g_{ij} which are scalars. As follows from theorem 2 such a concomitant is an algebraic concomitant of the tensors g_{ij} and R_{ijkl}. Thus it must satisfy the equation

$$\omega(\tilde{g}_{ij}, \tilde{R}_{ijkl}) = \omega(g_{ij}, R_{ijkl}). \tag{3.1}$$

In the case of a two-dimensional manifold X^2 the tensor R_{ijkl} has only one independent coordinate R_{1212}. This coordinate has the following transformation rule

$$\tilde{R}_{1212} = J^{-2} R_{1212}. \tag{3.2}$$

Thus it is a density of weight 2. Therefore equation (3.1) will have the form

$$\sigma(\tilde{g}_{ij}, \tilde{R}_{1212}) = \sigma(g_{ij}, R_{1212}). \tag{3.3}$$

We denote by S the number of positive signes in the canonical form of the tensor g_{ij}. The number S will be called the signature of the tensor g_{ij}.

We prove the following theorem:

THEOREM 3. *Every scalar differential concomitant of second order of a symmetric and regular tensor g_{ij} for $n=2$ is an arbitrary function of Gauss's curvature K and of the signature S.*

Proof. The proof is carried independently for the following cases: $S=2$, $S=1$, $S=0$.

CASE $S=2$. It is known that we may always find a non-singular matrix $||B^i_k||$ such that

$$||\tilde{g}_{ik}|| = ||B^s_i B^t_k g_{st}|| = \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix} \tag{3.4}$$

(cf. [3], p. 269).

We know that $g = \mathrm{Det}\,||g_{ij}||$ has the following transformation rule

$$\tilde{g} = J^{-2} g \tag{3.5}$$

where $J = \mathrm{Det}\,||A^i_k||$ and $||A^i_k|| = ||B^i_k||^{-1}$. It follows from equation (3.4) that the determinant J of the matrix $||A^i_k||$ satisfies the equation

$$J^2 = g. \tag{3.6}$$

Let us insert relations (3.4) and (3.6) into equation (3.3). We obtain

$$\sigma(g_{ik}, R_{1212}) = \sigma\left(1, 0, 0, 1, \frac{R_{1212}}{g}\right). \tag{3.7}$$

The value R_{1212}/g is GAUSS's curvature K of the space X^2 (cf. [5], p. 463):

$$K = \frac{R_{1212}}{g}. \tag{3.8}$$

Thus

$$\sigma(g_{ij}, R_{1212}) = \varrho(S, K).$$

CASE $S=1$. There exists a non-singular matrix $||B^i_k||$ such that

$$||\tilde{g}_{ij}|| = ||B^s_i B^t_j g_{st}|| = \begin{Vmatrix} 1 & 0 \\ 0 & -1 \end{Vmatrix}. \tag{3.9}$$

The determinant J of the matrix $\|A_k^i\| = \|B_k^i\|^{-1}$ satisfies the equation

$$J^2 = -g. \tag{3.10}$$

We insert relations (3.9) and (3.10) into (3.3). Then we obtain the relations

$$\sigma(g_{ij}, R_{1212}) = \sigma\left(1, 0, 0, -1, -\frac{R_{1212}}{g}\right). \tag{3.11}$$

It follows from (3.8) that

$$\sigma(g_{ij}, R_{1212}) = \eta(S, K).$$

CASE $S=0$. The proof is analogous to that in the case $S=2$.

This completes the proof.

We consider an arbitrary J-object ω with one component (cf. [1], p. 47). Its transformation rule has the form

$$\tilde{\omega} = \varphi(J)\,\omega, \tag{3.12}$$

where $\varphi(x)$ is a function satisfying the functional equation

$$\varphi(xy) = \varphi(x)\varphi(y). \tag{3.13}$$

We prove the following lemma:

LEMMA 1. *If a J-object with one component is a differential concomitant of the second order of the tensor g_{ij}, then the function φ in* (3.12) *satisfiies the condition*

$$\varphi(J) = \varphi(|J|).$$

Proof. Let ω be an arbitrary J-object with one component which is a differential concomitant of second order of the tensor g_{ij}. By a similar argument as in the proof of Theorem 3 we obtain

$$\omega = \varphi\left(\frac{1}{\sqrt{|g|}}\right)\psi(S, K), \tag{3.14}$$

where $\psi(S, K)$ is an arbitrary scalar. It follows from (3.14) that

$$\tilde{\omega} = \varphi\left(\frac{1}{\sqrt{|\tilde{g}|}}\right)\psi(S, K) = \varphi(|J|)\,\varphi\left(\frac{1}{\sqrt{|g|}}\right)\psi(S, K). \tag{3.15}$$

On the other hand, we have

$$\tilde{\omega} = \varphi(J)\,\omega = \varphi(J)\,\varphi\left(\frac{1}{\sqrt{|g|}}\right)\psi(S, K). \tag{3.16}$$

From (3.15) and (3.16) we have

$$\varphi(J) = \varphi(|J|). \tag{3.17}$$

Now we prove the following theorem:

THEOREM 4. *There do not exist differential concomitants of the second order of the symmetric and regular tensor g_{ij} for $n=2$ which are G-densities of a weight p.*

Proof. For a G-density of weight p the function φ has the form

$$\varphi(J) = (\operatorname{sgn} J)\,|J|^{p}. \tag{3.18}$$

Such a function φ does not satisfy condition (3.18).

THEOREM 5. *Every differential concomitant of the second order of the symmetric and regular tensor g_{ij} for $n=2$ which is a W-density of weight p has the form*

$$\varrho = |g|^{-\frac{p}{2}}\gamma(S, K),$$

where $\gamma(S, K)$ is an arbitrary scalar concomitant of the second order of the tensor g_{ij}.

Proof. Let ϱ be an arbitrary W-density of weight p. The value

$$\frac{\varrho}{|g|^{-\frac{p}{2}}}$$

is a scalar. From Theorem 3 we have

$$\frac{\varrho}{|g|^{-\frac{p}{2}}} = \gamma(S, K), \tag{3.19}$$

where $\gamma(S, K)$ is a scalar differential concomitant of the second order of the tensor g_{ij}.

This completes the proof.

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UWAGI O KOMITANTACH RÓŻNICZKOWYCH TENSORA KOWARIANTNEGO

Streszczenie

W pierwszej części tego artykułu dowodzi się pewnych twierdzeń o komitantach różniczkowych pierwszego i drugiego rzędu nieosobliwego, symetrycznego tensora g_{ik}. Twierdzenia te są szczególnymi przypadkami ogólnego twierdzenia podanego przez J. A. SCHOUTENA. Metoda pokazanego tutaj dowodu opiera się na równaniach funkcyjnych i nie wymaga żadnych założeń o regularności rozważanych funkcji.

W drugiej części zostały wyznaczone komitanty różniczkowe drugiego rzędu tensora g_{ik}, które są skalarami i W-gęstościami. Wykazano również, że nie istnieją komitanty różniczkowe drugiego rzędu tensora g_{ik}, które są G-gęstościami. Rozważania drugiej części prowadzone są dla $n = 2$.

Oddano do Redakcji 15 lipca 1969 r.